SCHRIFTELIJK TENTAMEN

ASTROPHYSICAL HYDRODYNAMICS

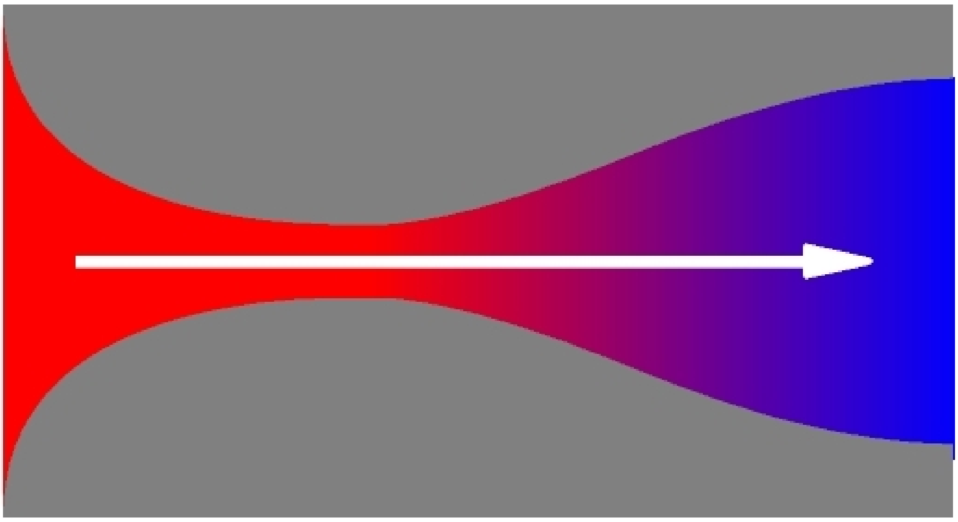
**$3^{rd}$ quarter 2015/2016**

April 6, 2016

NOTE: THIS EXAM CONTAINS 4 QUESTIONS, 6 pages.
Please assure you have read all pages and questions.
Mention your name and studentnr. on ALL pages that you hand in.

## Question 1.: Jets and the De Laval Nozzle

A very interesting application of the Bernoulli equation, for compressible fluids, concerns the *de Laval Nozzle.* A de Laval nozzle is a tube that is pinced in the middle, making a carefully balanced, asymmetric hourglass shape. The nozzle was developed in 1888 by the Swedish inventor Gustaf de Laval for use on a steam turbine. The principle was first used for rocket engines by Robert Goddard. An illustration of a de Laval Nozzle is shown in figure 1.



Figuur 1: Illustration of the de Laval Nozzle

a. We make the approximation of steady, quasi-1-D barotropic flow. Essential is that the flow is compressible (ie. not incompressible). The

1-D flow velocity (along the x-axis) is $u$, the density is $\rho$, the pressure $p$. Write the Bernoulli equation for compressible flow (ignoring an external force like gravity).

b. If the local sectional area of the nozzle is A, write the continuity equation.

c. Infer from the Bernoulli equation that

$$\frac{d\rho}{\rho} = -M^2 \frac{du}{u} \tag{1}$$

where $M = u/c_s$ is the Mach number of the flow, the ratio of flow velocity to the sound speed,

$$c_s^2 = \frac{dp}{d\rho} \,. \tag{2}$$

d. Invoking the continuity equation (question b), show that

$$\frac{d\rho}{\rho} + \frac{du}{u} + \frac{dA}{A} = 0 \tag{3}$$

e. and hence show that

$$(1 - M^2)\frac{du}{u} = -\frac{dA}{A} \,. \tag{4}$$

f. Investigating the consequences of this nozzle equation, describe first what the consequence is for the flow velocity when the cross section $A$ changes and the flow is subsonic. On the other hand, what happens if the flow is supersonic ? Why is the latter at first counterintuitive ? How can this be explained when looking at the development of the density $\rho$ ?

g. A sonic transition happens when the flow passes from subsonic to supersonic, ie. when $M = 1$. If $du/dx$ is finite, why does this happen at the throat of the nozzle ?

**Question 2.: Incompressible inviscid flow**

Consider a source (or a sink) in three dimensions. The flow field must point radially outward from origin and thus $\vec{u} = f(r)\hat{r}$.

a) Using spherical polar coordinates show that, if the flow is incompressible then the velocity field is given by,

$$\vec{u} = \frac{m}{4\pi r^2}\hat{r} \tag{5}$$

where $m$ is the strength of the source given by,

$$m = \int_S \vec{u} \cdot \hat{n} dS \tag{6}$$

b) Now apply Euler's equation for a steady flow and show that pressure follows

$$p = -\frac{m^2\rho}{32\pi^2 r^4} + \mathbf{const} \tag{7}$$

where $\rho$ is the density.

c) Now we know that the potential $\phi$ and stream function $\psi$ are defined as follows in polar $(r, \theta)$ co-ordinates:

$$\begin{aligned} u_r &= \frac{\partial \phi}{\partial r} = \frac{1}{r^2 \sin\theta} \frac{\partial \psi}{\partial \theta} \\ u_\theta &= \frac{1}{r}\frac{\partial \phi}{\partial \theta} = -\frac{1}{r \sin\theta}\frac{\partial \psi}{\partial r} \end{aligned} \tag{8}$$

Show that in this case,

$$\begin{aligned} \phi &= -\frac{m}{4\pi r} \\ \psi &= \frac{m}{4\pi}(1 - \cos\theta) \end{aligned} \tag{9}$$

## Question 3.: Hydrostatics

In case of static fluids Euler's equation becomes

$$\boldsymbol{\nabla} P = \rho \mathrm{g}\,, \tag{10}$$

where $P$ is the pressure of the fluid and $\rho$g the force acting on the fluid.

a) Consider a very large, very massive fluid such that self-gravity becomes important. The Poisson equation relates the gravitational potential $\phi$ to the density $\rho$

$$\nabla^2 \phi(\boldsymbol{r}) = 4\pi G_N \rho(\boldsymbol{r})\,, \tag{11}$$

where $G_N$ is Newton's constant. Use the assumption of spherical symmetry to find,

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -4\pi G_N \rho\,. \tag{12}$$

b) In order to solve this, we need a relation between $P$ and $\rho$, called the *equation of state*. Often this is a polytropic relation

$$P \propto \rho^{1+1/n}\,, \tag{13}$$

where $n$ is called the polytropic index. Show that for the gaseous planets, where $P = \alpha\rho^2$ the above differential equation turns into the following form

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \rho}{\partial r}\right) + \beta^2 \rho = 0\,. \tag{14}$$

where $\beta^2 = \frac{2\pi G_N}{\alpha}$.

c) The above equation has a solution,

$$\rho(r) = A\frac{\sin(\beta r)}{\beta r} + B\frac{\cos(\beta r)}{\beta r} \tag{15}$$

$A, B$ are constants. Using proper boundary conditions show that the central pressure in the gaseous planets is,

$$P(0) = \frac{\pi G M^2}{8R^4} \tag{16}$$

d) Assume a planet with a constant density (no relation between $P$ and $\rho$) has radius $R$ and mass $M$. The pressure at the center is $P_c$, and the pressure at the surface is $P_R = 0$. Derive an expression for the pressure $P(r)$ as function of radius in terms of $M$ and $R$ (and $G$ etc.). What is the central pressure?

## Question 4.: Navier-Stokes Equation and Poiseuille Equation

We are going to investigate the Navier-Stokes equation, and its application towards the flow through a cylindrical pipe.

a) Write down the Navier-Stokes equation for an incompressible fluid with viscosity $\eta$, with $\vec{v}$ the velocity of the flow, $\rho$ its density and $p$ its pressure.

b) A characteristic of viscous fluids is the Reynolds number. Specify the definition of the Reynolds number, and derive an estimate for its value for a fluid with typical velocity $U$ and size $L$. For which values of the Reynolds number the viscous effects beecome important ?

One situation in which we can derive an analytical expression for a viscous flow is that of a steady, laminar flow through a tube or pipe with radius $R$. Steady means that the time derivative of the relevant physical quantities is zero: $\partial/\partial t = 0$. For the pipe we use cylindrical coordinates, $x$ along the length of the pipe, radius $r$ and sectional angle $\theta$.

The flow is only along the length of the pipe, $u_x = v$. The radial and angular components of the fluid velocity are zero: $u_r = u_{theta} = 0$. The flow is axisymmetric, and can only vary in the radial direction or, possibly, along the x-direction. That is $\partial/\partial\theta = 0$. The flow is fully developed and will not vary along the x-direction as the diameter of the pipe is constant along its length:

$$\frac{\partial v}{\partial x} = 0 \tag{17}$$

c) Show that the Navier-Stokes equation for this situation can be written as

$$\begin{aligned} \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right) &= \frac{1}{\eta}\frac{\partial p}{\partial x} \\ 0 &= \frac{1}{\eta}\frac{\partial p}{\partial r} \end{aligned} \tag{18}$$

d) Argue that the pressure drop along the $x$-direction in the pipe is linear, i.e. that $(\partial p/\partial x)$ is constnat, and thus

$$\frac{\partial p}{\partial x} = -\frac{\Delta p}{L} \tag{19}$$

where $\Delta p$ is the pressure drop along a length $L$ of the pipe.

e) Subsequently, show that the general solution for this equation can be written as

$$v = \frac{1}{4\eta}\frac{\partial p}{\partial x}\,r^2 \,+\, c_1 \ln r \,+\, c_2 \tag{20}$$

where $c_1$ and $c_2$ are integration constants.

f) From the boundary condition that the velocity $v$ is finite at $r = 0$, it follows that $c_1 = 0$. Show that from the no slip boundary condition at the wall of the pipe, ie. $u_x = v = 0$ at $r = R$ follows:

$$c_2 = -\frac{1}{4\eta}\frac{\partial p}{\partial x}\,R^2 \tag{21}$$

g) and that therefore the generic solution for the flow field is given by

$$v(r) = \frac{1}{4\eta}\frac{\Delta p}{L}\,(R^2 - r^2)\,. \tag{22}$$

This is called the Hagen-Poiseuille equation. Describe and explain how the flow field $v(r)$ in the pipe behaves as function of radius $r$.

h) Show that for a fluid with a density $\rho$ the amount of mass transported through the pipe, per time interval, is given by

$$\dot{M} \;=\; \int_0^R 2\pi\rho v r \mathrm{d}r \;=\; \frac{\pi\Delta p}{4\eta L}\,R^4\,. \tag{23}$$

SUCCES !!!!
BEDANKT VOOR JULLIE AANDACHT EN INTERESSE !!!!

Rien & Saikat